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Methods for Solving Some Fractional Integral

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Abstract: **In this paper, based on Jumarie's modified Riemann-Liouville (R-L) fractional calculus, we evaluate some fractional integral. The main methods we used are fractional L'Hospital's rule, integration by parts for fractional calculus, and a new multiplication of fractional analytic functions. In fact, our result is the generalization of classical calculus result.**

Keywords: **Jumarie's modified R-L fractional calculus, fractional integral, fractional L'Hospital's rule, integration by parts for fractional calculus, new multiplication, fractional analytic functions.**

I. INTRODUCTION

Fractional calculus is a natural extension of the traditional calculus. In fact, since the beginning of the theory of differential and integral calculus, some mathematicians have studied their ideas on the calculation of non-integer order derivatives and integrals. During the 18th and 19th centuries, there were many famous scientists such as Euler, Laplace, Fourier, Abel, Liouville, Grunwald, Letnikov, Riemann, Laurent, Heaviside, and some others who reported interesting results within fractional calculus.

In recent years, fractional calculus has become an increasingly popular research area due to its effective applications in different scientific fields such as economics, viscoelasticity, physics, mechanics, biology, electrical engineering, control theory, and so on [1-9]. However, the definition of fractional derivative is not unique. The commonly used definitions include Riemann-Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald-Letnikov (G-L) fractional derivative, and Jumarie's modified R-L fractional derivative [10-13]. Since Jumarie type of R-L fractional derivative helps to avoid non-zero fractional derivative of constant function, it is easier to use this definition to connect fractional calculus with traditional calculus.

In this paper, based on Jumarie type of R-L fractional calculus, we evaluate the following α-fractional integral:

$$\left({}_0I_x^\alpha\right)\left[Ln_\alpha(x^\alpha)\otimes Ln_\alpha\left(1-\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)\right], \tag{1}$$

where $0<\alpha\le 1$, and $0<\frac{1}{\Gamma(\alpha+1)}x^\alpha<1$. Fractional L'Hospital's rule, integration by parts for fractional calculus, and a new multiplication of fractional analytic functions play important roles in this paper. In fact, our result is the generalization of traditional calculus result.

II. PRELIMINARIES

Firstly, the fractional derivative used in this paper and its properties are introduced below.

Definition 2.1 ([14])**:** Assume that $0<\alpha\le 1$, and x_0 is a real number. The Jumarie's modified Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{x_0}D_x^\alpha\right)[f(x)]=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^x\frac{f(t)-f(x_0)}{(x-t)^\alpha}dt\,. \tag{2}$$

And the Jumarie type of Riemann-Liouville α-fractional integral is defined by

$$\left({}_{x_0}I_x^{\alpha}\right)[f(x)]=\frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}\frac{f(t)}{(x-t)^{1-\alpha}}dt\,, \tag{3}$$

where $\Gamma(\ \ )$ is the gamma function.

Proposition 2.2 ([15])**:** *If* α,β,x_0,C *are real numbers and* $\beta\geq\alpha>0$, *then*

$$\left({}_0D_x^{\alpha}\right)\left[x^{\beta}\right]=\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}x^{\beta-\alpha}, \tag{4}$$

and

$$\left({}_0D_x^{\alpha}\right)[C]=0. \tag{5}$$

Next, we introduce the definition of fractional analytic function.

Definition 2.3 ([16])**:** Assume that x and a_k are real numbers for all k, and $0<\alpha\leq 1$. If the function $f_\alpha:[a,b]\rightarrow R$ can be expressed as an α-fractional power series, i.e., $f_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}x^{k\alpha}$, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic function.

In the following, we introduce a new multiplication of fractional analytic functions.

Definition 2.4 ([17])**:** If $0<\alpha\leq 1$. Suppose that $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are two α-fractional analytic functions,

$$f_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}x^{k\alpha}=\sum_{k=0}^{\infty}\frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes k}, \tag{6}$$

$$g_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}x^{k\alpha}=\sum_{k=0}^{\infty}\frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes k}. \tag{7}$$

Then we define

$$\begin{aligned}&f_\alpha(x^\alpha)\otimes g_\alpha(x^\alpha)\\&=\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}x^{k\alpha}\otimes\sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}x^{k\alpha}\\&=\sum_{k=0}^{\infty}\frac{1}{\Gamma(k\alpha+1)}\left(\sum_{m=0}^{k}\binom{k}{m}a_{k-m}b_m\right)x^{k\alpha}.\end{aligned} \tag{8}$$

Equivalently,

$$\begin{aligned}&f_\alpha(x^\alpha)\otimes g_\alpha(x^\alpha)\\&=\sum_{k=0}^{\infty}\frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes k}\otimes\sum_{k=0}^{\infty}\frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes k}\\&=\sum_{k=0}^{\infty}\frac{1}{k!}\left(\sum_{m=0}^{k}\binom{k}{m}a_{k-m}b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes k}.\end{aligned} \tag{9}$$

Definition 2.5 ([17])**:** Suppose that $0<\alpha\leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are α-fractional analytic functions,

$$f_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}x^{k\alpha}=\sum_{k=0}^{\infty}\frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes k}, \tag{10}$$

$$g_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}x^{k\alpha}=\sum_{k=0}^{\infty}\frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes k}. \tag{11}$$

The compositions of $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are defined by

$$(f_\alpha\circ g_\alpha)(x^\alpha)=f_\alpha\left(g_\alpha(x^\alpha)\right)=\sum_{k=0}^{\infty}\frac{a_k}{k!}\left(g_\alpha(x^\alpha)\right)^{\otimes k}, \tag{12}$$

and

$$(g_\alpha\circ f_\alpha)(x^\alpha)=g_\alpha\left(f_\alpha(x^\alpha)\right)=\sum_{k=0}^{\infty}\frac{b_k}{k!}\left(f_\alpha(x^\alpha)\right)^{\otimes k}. \tag{13}$$

Definition 2.6 ([17])**:** Let $0<\alpha\leq 1$. If $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions satisfies

$$(f_\alpha\circ g_\alpha)(x^\alpha)=(g_\alpha\circ f_\alpha)(x^\alpha)=\frac{1}{\Gamma(\alpha+1)}x^{\alpha}. \tag{14}$$

Then $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are called inverse functions of each other.

Some fractional analytic functions are introduced below.

Definition 2.7 ([18])**:** If $0 < \alpha \le 1$, and x, x_0 are real numbers. The α-fractional exponential function is defined by

$$E_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{x^{k\alpha}}{\Gamma(k\alpha+1)} = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes k}. \tag{15}$$

And the α-fractional logarithmic function $Ln_\alpha(x^\alpha)$ is the inverse function of $E_\alpha(x^\alpha)$.

Theorem 2.8 (integration by parts for fractional calculus) ([19])**:** *Assume that* $0 < \alpha \le 1$, a, b *are real numbers, and* $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ *are* α*-fractional analytic functions, then*

$$\left({}_aI_b^\alpha\right)\left[f_\alpha(x^\alpha) \otimes \left({}_aD_x^\alpha\right)[g_\alpha(x^\alpha)]\right] = [f_\alpha(x^\alpha) \otimes g_\alpha(x^\alpha)]_{x=a}^{x=b} - \left({}_aI_b^\alpha\right)\left[g_\alpha(x^\alpha) \otimes \left({}_aD_x^\alpha\right)[f_\alpha(x^\alpha)]\right]. \tag{16}$$

Theorem 2.9 ([20]) (fractional L'Hospital's rule)**:** *Suppose that* $0 < \alpha \le 1$, c *is a real number, and* $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$, $[g_\alpha(x^\alpha)]^{\otimes -1}$ *are* α*-fractional analytic functions at* $x = c$. *If* $\lim_{x\to c} f_\alpha(x^\alpha) = \lim_{x\to c} g_\alpha(x^\alpha) = 0$, *or* $\lim_{x\to c} f_\alpha(x^\alpha) = \pm\infty$, *and* $\lim_{x\to c} g_\alpha(x^\alpha) = \pm\infty$. *Assume that* $\lim_{x\to c} f_\alpha(x^\alpha) \otimes [g_\alpha(x^\alpha)]^{\otimes -1}$ *and* $\lim_{x\to c}\left({}_cD_x^\alpha\right)[f_\alpha(x^\alpha)] \otimes \left[\left({}_cD_x^\alpha\right)[g_\alpha(x^\alpha)]\right]^{\otimes -1}$ *exist,* $\left({}_cD_x^\alpha\right)[g_\alpha(x^\alpha)](c) \ne 0$. *Then*

$$\lim_{x\to c} f_\alpha(x^\alpha) \otimes [g_\alpha(x^\alpha)]^{\otimes -1} = \lim_{x\to c}\left({}_cD_x^\alpha\right)[f_\alpha(x^\alpha)] \otimes \left[\left({}_cD_x^\alpha\right)[g_\alpha(x^\alpha)]\right]^{\otimes -1}. \tag{17}$$

III. MAIN RESULT

In this section, we find some fractional integral in this article. Firstly, two lemmas are needed.

Lemma 3.1: *Let* $0 < \alpha \le 1$, *and* $-1 < \frac{1}{\Gamma(\alpha+1)}x^\alpha < 1$, *then*

$$Ln_\alpha\left(1 - \frac{1}{\Gamma(\alpha+1)}x^\alpha\right) = -\sum_{k=0}^{\infty} \frac{1}{k+1}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes(k+1)}. \tag{18}$$

Proof Since $-1 < \frac{1}{\Gamma(\alpha+1)}x^\alpha < 1$, it follows that

$$\left(1 - \frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes -1} = \sum_{k=0}^{\infty}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes k}. \tag{19}$$

And hence,

$$\begin{aligned}
&\left({}_0I_x^\alpha\right)\left[\left(1 - \frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes -1}\right] \\
&= \left({}_0I_x^\alpha\right)\left[\sum_{k=0}^{\infty}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes k}\right] \\
&= \sum_{k=0}^{\infty}\left({}_0I_x^\alpha\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes k}\right] \\
&= \sum_{k=0}^{\infty} \frac{1}{k+1}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes(k+1)}.
\end{aligned} \tag{20}$$

Therefore,

$$\begin{aligned}
&Ln_\alpha\left(1 - \frac{1}{\Gamma(\alpha+1)}x^\alpha\right) \\
&= -\left({}_0I_x^\alpha\right)\left[\left(1 - \frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes -1}\right] \\
&= -\sum_{k=0}^{\infty} \frac{1}{k+1}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes(k+1)}.
\end{aligned}$$

Q.e.d.

Lemma 3.2: *If* $0 < \alpha \le 1$, *and* k *is a non-negative integer, then*

$$\left({}_0I_x^\alpha\right)\left[Ln_\alpha(x^\alpha) \otimes \left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes(k+1)}\right] = Ln_\alpha(x^\alpha) \otimes \frac{1}{k+2}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes(k+2)} - \frac{1}{(k+2)^2}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes(k+2)}. \tag{21}$$

Proof By fractional L'Hospital's rule and integration by parts for fractional calculus,

$$\left({}_0I_x^\alpha\right)\left[Ln_\alpha(x^\alpha)\otimes\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes(k+1)}\right]$$

$$=\left({}_0I_x^\alpha\right)\left[Ln_\alpha(x^\alpha)\otimes\left({}_0D_x^\alpha\right)\left[\frac{1}{k+2}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes(k+2)}\right]\right]$$

$$=\left[Ln_\alpha(x^\alpha)\otimes\frac{1}{k+2}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes(k+2)}\right]_0^x-\left({}_0I_x^\alpha\right)\left[\frac{1}{k+2}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes(k+1)}\right]$$

$$=Ln_\alpha(x^\alpha)\otimes\frac{1}{k+2}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes(k+2)}-\frac{1}{(k+2)^2}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes(k+2)}.\qquad \text{Q.e.d.}$$

Theorem 3.3: *Let* $0<\alpha\leq 1$, *and* $0<\frac{1}{\Gamma(\alpha+1)}x^\alpha<1$, *then the* α*-fractional integral*

$$\left({}_0I_x^\alpha\right)\left[Ln_\alpha(x^\alpha)\otimes Ln_\alpha\left(1-\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)\right]$$

$$=-Ln_\alpha(x^\alpha)\otimes\sum_{k=0}^{\infty}\frac{1}{(k+1)(k+2)}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes(k+2)}+\sum_{k=0}^{\infty}\frac{1}{(k+1)(k+2)^2}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes(k+2)}.\qquad(22)$$

Proof $\left({}_0I_x^\alpha\right)\left[Ln_\alpha(x^\alpha)\otimes Ln_\alpha\left(1-\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)\right]$

$$=\left({}_0I_x^\alpha\right)\left[Ln_\alpha(x^\alpha)\otimes-\sum_{k=0}^{\infty}\frac{1}{k+1}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes(k+1)}\right]\quad\text{(by Lemma 3.1)}$$

$$=-\sum_{k=0}^{\infty}\frac{1}{k+1}\left({}_0I_x^\alpha\right)\left[Ln_\alpha(x^\alpha)\otimes\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes(k+1)}\right]$$

$$=-\sum_{k=0}^{\infty}\frac{1}{k+1}\left[Ln_\alpha(x^\alpha)\otimes\frac{1}{k+2}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes(k+2)}-\frac{1}{(k+2)^2}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes(k+2)}\right]\quad\text{(by Lemma 3.2)}$$

$$=-Ln_\alpha(x^\alpha)\otimes\sum_{k=0}^{\infty}\frac{1}{(k+1)(k+2)}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes(k+2)}+\sum_{k=0}^{\infty}\frac{1}{(k+1)(k+2)^2}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes(k+2)}.\qquad\text{Q.e.d.}$$

IV. CONCLUSION

In this paper, based on Jumarie type of R-L fractional calculus, we solve some fractional integral. A new multiplication of fractional analytic functions plays an important role in this article. The main methods we used are fractional L'Hospital's rule, and integration by parts for fractional calculus. In fact, our result is the generalization of the result in ordinary calculus. In the future, we will continue to use the above methods to study the problems in fractional differential equations and applied mathematics.

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